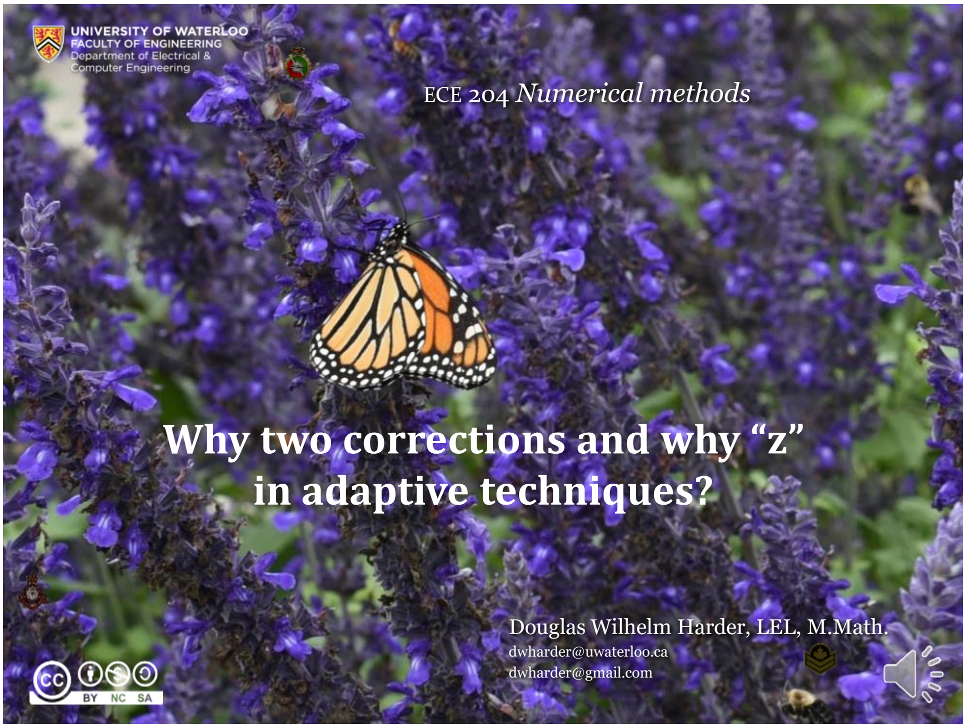


1

Why two corrections and why "z" for adaptive techniques?

Introduction

- In this topic, we will
  - Discuss why we use $2|z - y|$ and $h \leftarrow 0.9ah$ with each step
  - Discuss why we use z when the error estimates are for y

2

2

Two corrections

- For both adaptive Euler-Heun and Dormand-Prince methods, you will recall that there are two corrections
  - First, we claimed that because the error of y is $O(h^n)$ and the error of z is $O(h^{n+1})$, we could estimate the error of y from the true solution by calculating $|z - y|$
    - To ensure that we are not underestimating the error of y, we doubled this estimate to $2|z - y|$
  - Second, once we calculate the new scaling factor ah, instead of using ah, we use $0.9ah$ as the new scaling factor
- Why is this second correction necessary?
  - After all, we're already likely over-estimating the error of y…

The scaling factor and $2|z - y|$

- When we calculate a, we are looking for the ideal step size h that will give us the acceptable error $\varepsilon_{abs}h$
  - Issue: in calculating a, we are already using $2|z - y|$
  - This, of course, will not be perfect, as everything is an approximation…

Too small a step size

- Suppose we approximate the next value with a given step-size h and we calculate a and find $a = 0.831 < 1$
  - This indicates that we must recalculate the next value, but now using a smaller step size $0.831h$
  - Problem: assuming this is close to the ideal step size, it follows that when we calculate the new a, we will have $a \approx 1$
  - Problem: if $a \approx 1$, then there is a 50-50 chance that $a < 1$ again
    - Thus, half the time, we will have to recalculate this step again
- The worst-case scenario is we continue to recalculate the current step and get a sequence of scaling factors

$$0.831, 0.9175, 0.9938, 0.9987, \ldots$$

  - We may never get past the current step!

Too large a step size

- Similarly, suppose we approximate the next value with a given step-size h and we calculate a and find $a = 1.238 > 1$
  - This means we can use the current approximation, and we proceed to next step but using $1.238h$
  - Problem: again, assuming this is close to the ideal step size, it follows that when we calculate the new a for the next step we will have $a \approx 1$
  - Problem: if $a \approx 1$ for the next step, then there is a 50-50 chance that $a < 1$ again
    - Thus, half the time, we will have to recalculate the next step

Using 0.9h

- Instead, by using 0.9h as the next step size,
  this seeks to ensure that the next $a \approx 0.9^{-1} = 1.11$
  - This seeks to ensure that while our step size is smaller than necessary, it also ensures that we do not have to recalculate
- Consequence: with this approach, we will have to calculate approximately 11% more steps than should be necessary, but we will very seldom need to recalculate any one step

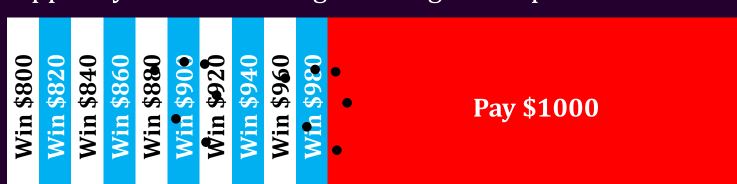
7

7

Why two corrections and why "z" for adaptive techniques?

An analogy

- Suppose you are throwing a dart against a painted wall



  - We will assume you are the average person
    - Otherwise, just make the strips narrower… ☺
  - You could aim for the "Win $980" stripe,
    but if you erred to the right, you'd lose $1000
  - Instead, if you aimed for the "Win $900"
    - If you erred slightly to the right, you'd still win
  - That is why we use 0.9ah

8

8

Why two corrections and why "z" for adaptive techniques?

Why "z"?

- We discussed how the step size is chosen to ensure that the error of the approximation *y* is no greater than $\varepsilon_{\text{abs}} h$
  - Question: if we are creating an upper bound for *y*, why are we using *z*?
  - Answer: its subtle…
    - Generally, *z* will be more accurate than *y* anyways
    - Dormand-Prince chooses coefficients to minimize the coefficients of the error term for *z*
  - Question: If using "z" usually gives us a better approximation, can't we use a larger step size?
  - Answer: perhaps, but we have no estimate for the error of "z"
    - We would have no assurance the error is appropriate constrained
    - For fun: implement the adaptive techniques, but use "y"

9

9

Why two corrections and why "z" for adaptive techniques?

Summary

- Following this topic, you now
  - Understand that $2|z - y|$ ensures we are likely to overestimate the error of *y*
  - Understand that the calculated scaling factor *a* ensures that with the next step, $a \approx 1$
    - Realize that this means there is a 50-50 chance we may have to recalculate with a smaller step size…
  - Understand that instead, we reduce the chance of having to recalculate to a very small likelihood with the cost being only approximately 11% more steps than is necessary
  - Understand that the error estimate is for *y*, but we use *z* as it is more accurate, but we cannot increase the step size because we have no guarantee on the error of *z*

10

10

References

[1] https://en.wikipedia.org/wiki/Dormand-Prince_method

Why two corrections and why "z" for adaptive techniques?

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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